Tropical Geometry and Mechanism Design

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Section 1

Mechanisms and Tropical Linear Algebra of Incentive Compatibility

Economic Model - Mechanisms

Setup: $m \in \mathbb{N}$ **outcomes**, there is a **single player** with **type-space** $T \subset \mathbb{R}^m$.

The player has a true type $t \in T$, determined by nature.
Coordinate t_j measures the value of outcome j.
Only the agent knows t, "private information".

Definition

A **mechanism** is a pair (g, p) with an outcome function $g : T \to [m]$, and a payment vector $p \in \mathbb{R}^m$.

Incentive Compatibility

A mechanism (g, p) **induces a game**.

- Strategy is a report: $t \to r(t)$, with $r(t) \in T$
- Game's outcome: $g(r) \in [m]$ and a payment $p_{g(r)} \in \mathbb{R}$.

The agent chooses a strategy to maximize his **utility**

$$r(t) \in \arg\max_{r \in T} \{(t - p)_{g(r)}\} \text{ for all } t \in T.$$

Definition

A mechanism (g, p) is **incentive compatible** (IC) if

$$t_{g(t)} - p_{g(t)} \geq t_{g(r)} - p_{g(r)}$$

for all $t, r \in T$.

Tropical Linear Algebra of IC

$(\mathbb{R}, \underline{\oplus}, \odot)$ min-plus semi-ring: $a \underline{\oplus} b = \min\{a, b\}$, $a \odot b = a + b$.

Given an outcome function $g : T \to [m]$, define the matrix L^g:

$$L^g_{ij} = \inf_{t \in g^{-1}(i)} \{t_i - t_j\} \quad \text{for all } i, j \in [m].$$

$(\underline{\lambda}(L^g), p) \in \mathbb{R} \times \mathbb{R}^m$ is a **min-plus eigenpair** if

$$L^g \underline{\odot} p = \underline{\lambda}(L^g) \underline{\odot} p.$$

Theorem (Rochet, Cuninghame-Green)

A mechanism (g, p) is IC if and only if $L^g \underline{\odot} p = p$. In other words, (g, p) is IC if and only if $(0, p)$ is an eigenpair of L^g.

Section 2

Geometry of Mechanisms

In the following assume that $|T| = r < \infty$ and 'generic'.

$(\mathbb{R}, \underline{\oplus}, \odot)$ min-plus semi-ring: $a \underline{\oplus} b = \min\{a, b\}$, $a \odot b = a + b$.
$(\mathbb{R}, \overline{\oplus}, \odot)$ max-plus semi-ring: $a \overline{\oplus} b = \max\{a, b\}$, $a \odot b = a + b$.

Tropical Convex Geometry I

Tropical affine space $\mathbb{TP}^{m-1} := \mathbb{R}^m / \sim$ with

$$(x_1, \ldots, x_m) = x \sim x \odot a = (x_1 + a, \ldots, x_m + a)$$

Min-plus hyperplane with apex t is

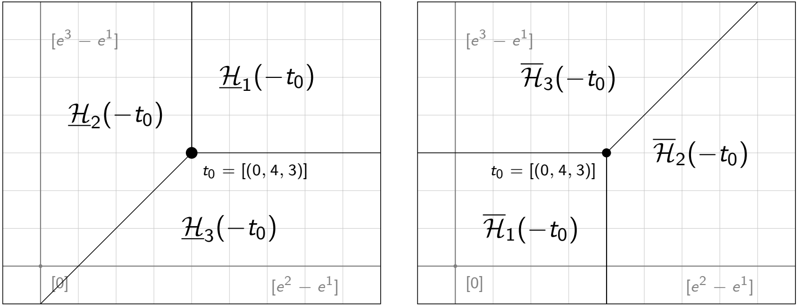
$$\underline{\mathcal{H}}(-t) = \left\{ q \in \mathbb{TP}^{m-1} \ : \ \exists i \neq j \text{ with } q_i - t_i = q_j - t_j = \min_{k \in [m]} \{q_k - t_k\} \right\},$$

For a subset $I \subset [m]$, the **halfspace of type** I with apex t is

$$\underline{\mathcal{H}}_I(-t) = \left\{ q \in \mathbb{TP}^{m-1} \ : \ q_i - t_i = \min_{k \in [m]} \{q_k - t_k\} \ \forall i \in I \right\}$$

$\underline{\mathcal{H}}(-T)$ is **tropical hyperplane arrangement**.

Tropical Convex Geometry II



(a) A min-plus hyperplane (b) A max-plus hyperplane

Figure : Tropical hyperplanes

Tropical Convex Geometry III

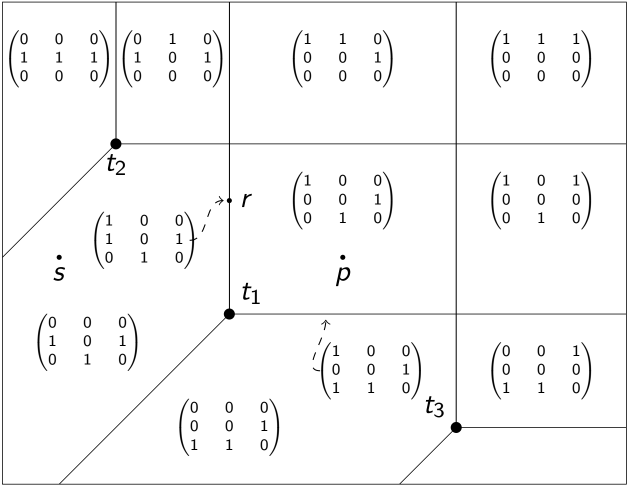


Figure : A min-plus arrangement on three points.

Covectors and Combinatorics of Arrangements

Covectors encode combinatorics of arrangement $\underline{\mathcal{H}}(-T)$, $\underline{\mathrm{coVec}}_T(q) \in \{0,1\}^{m\times r}$, with

$$\underline{\mathrm{coVec}}_T(q)_{it} = 1 \quad \Leftrightarrow \quad q \in \underline{\mathcal{H}}_i(-t).$$

For a bipartite graph g on $[m] \times T$, define

$$\begin{aligned} \mathcal{P}(g) := &\{q \in \mathbb{TP}^{m-1} \;:\; q \in \underline{\mathcal{H}}_i(-t) \text{ for all } (i,t) \in g\} \\ = &\{q \in \mathbb{TP}^{m-1} \;:\; t \in \overline{\mathcal{H}}_i(-q) \text{ for all } (i,t) \in g\} \end{aligned}$$

If $\mathcal{P}(g) \neq \emptyset$, then all points in relative interior have same covector $\nu(g)$, called the **basic covector** of g.

Basic cells

Definition

A covector ν is **basic** if $\nu = \nu(g)$ for some surjective outcome function. Call $\mathcal{P}(\nu)$ a **basic cell**.

Theorem

A surjective mechanism (g, p) is IC if and only if $p \in P$ for some basic cell P and $g \leq \underline{\text{coVec}}_T(P)$. Moreover, the cell $P(\nu(g))$ consists of all supporting payments.

Theorem

If T is finite and generic, then the basic cells are precisely the full-dimensional, bounded cells of $\underline{\mathcal{H}}(-T)$.

Geometric construction of Mechanisms

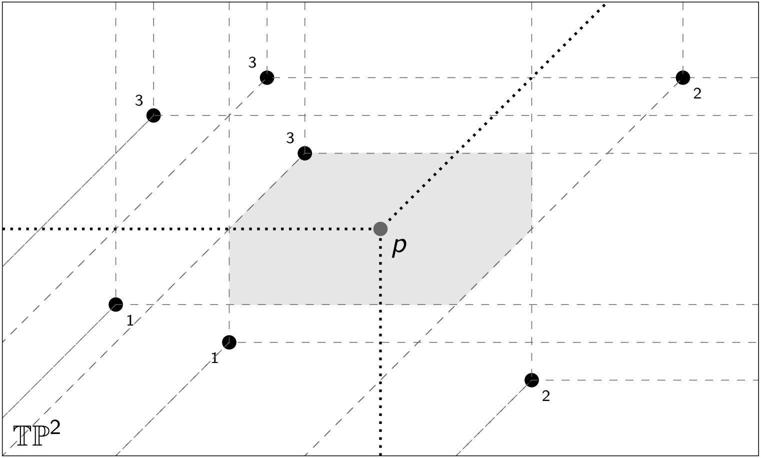


Figure : Geometric construction of an IC mechanism

Section 3

Geometry and Algebra of IC

Generators of the Eigenspace

Let $g : T \to [m]$ be IC. Recall

$$L^g_{ij} = \inf_{t \in g^{-1}(i)} \{t_i - t_j\} \quad \text{for all } i, j \in [m].$$

The tropical eigenspace is the set

$$\underline{\mathrm{Eig}}(L^g) = \{p \in \mathbb{TP}^{m-1} \;:\; L^g \underline{\odot} p = p\}$$

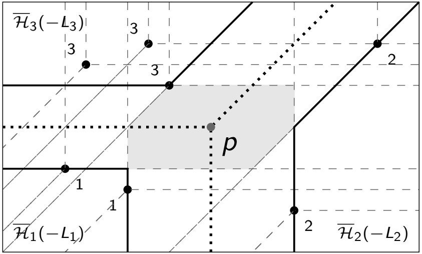
Define:

$$\underline{\mathrm{Eig}}_0(L^g) := \bigcap_{i=1}^{m} \underline{\mathcal{H}}_i(-L_{i,\bullet})$$

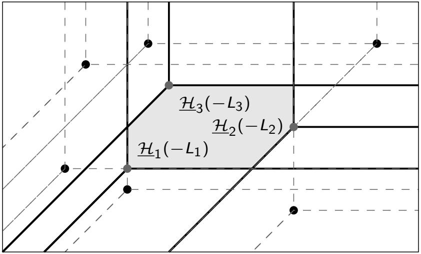
Theorem

If g is IC, then $\underline{\mathrm{Eig}}_0(L^g)$ is the set of IC payments.

Geometry of IC payments



(a) Generic types labeled by a mechanism



(b) The geometry of IC payments

Some Consequences

A mechanism (g, p) is **Revenue Equivalen** (RE) if there is a unique IC payments $p \in \mathbb{TP}^{m-1}$.

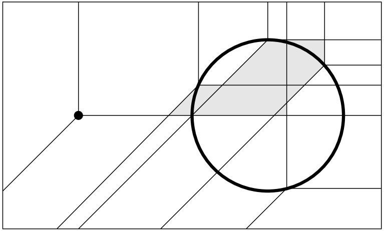
Classical property helpful to find revenue maximal mechanisms. Tropical geometry provides interpretation and another strategy.

- Tropical view of RE:
  - Geometrically: associated basic cell consists of exactly one point, C. and Tran 2016
  - Algebraically: pairwise tropical linear dependence.

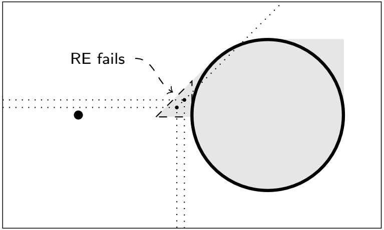
    (cf. Heydenreich et. al., Econometrica 2009)
- In the absence of RE: "extremal payments" (Kos and Messner, JET 2013) correspond to faces of basic cell supported by $[(0, -1, \ldots, -1)]$ and $[(0, 1, \ldots, 1)]$.
- Easy algorithm to find **revenue maximizing** mechanisms. Calculate expected revenue for each cell (using price in the face supported by $[(0, 1, \ldots, 1)]$). Then maximize over cells.

Geometry of RE

One can handle infinite and non-generic type-spaces via approximation.



(a) A finite approximation of T with five points and its min-plus arrangement.



(b) The basic cells T in gray.

Figure : Infinite type space via approximations

Further Results and Concludsion

Geometric theory of mechanism design and an algebraic understanding of IC payments.

Further results.

1. Full characterization of all IC mechanisms, for non-generic and infinite arrangements.
2. Extension to multiple players.
3. For a given type-space T, characterization of all matrices L that are generating sets of an IC mechanism supported on T.
4. Geometric criterion for the dimension of the Eigenspace.