# Ramification of Rough Paths

after

Massimiliano Gubinelli

presented by

Robert A. Crowell

University of Bonn

September 11 and 15, 2016

Prepared for the Hausdorff Summer School

*Paraproducts and Analysis of Rough Paths*

# Section 1

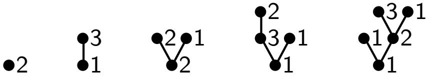
## Trees, Gardening and Forestry

# Decorated Rooted Trees

A **rooted tree** is a finite, cycle-free graph with a distinguished node (its root).

Let $\mathcal{L}$ be a finite set of labels. A $\mathcal{L}$-**decorated tree** is a rooted tree together with an association of a label to every vertex.

For example $\mathcal{L} = \{1, 2, 3\}$:



# Cultivating Trees

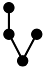
We have a recursive way of growing rooted, labeled trees.

Given $\tau_1, \ldots, \tau_k$ trees and a label $a \in \mathcal{L}$, let

$$\tau = [\tau_1, \cdots, \tau_k]_a$$

be the tree obtained by attaching $\tau_1, \cdots, \tau_k$ to a new root $a$.

*Observe that we can grow any tree using the set of labeled vertices $\{\bullet_a\}_{a\in\mathcal{L}}$ and the map $[-]_-$.*

$[\bullet] =$  $\quad [\bullet, [\bullet]] =$ 

# Tree Polynomial Algebra

Let $\mathcal{T}_{\mathcal{L}}$ denote the set of decorated trees and $\mathbf{1}$ the empty tree.

We can define the **tree polynomial algebra**

$$\mathcal{A}\mathcal{T}_{\mathcal{L}} = \langle \mathcal{T}_{\mathcal{L}} \cup \{\mathbf{1}\} \rangle_{\mathbb{R}-\mathsf{Alg}}$$

Elements are finite formal sums of formal monomials with coefficients in $\mathbb{R}$.

Explicit construction: $\mathcal{F}_{\mathcal{L}} = \{\tau_1 \cdots \tau_k \ : \ n \in \mathbb{N}_0, \tau_i \in \mathcal{T}_{\mathcal{L}}\}$ is the set of labeled **forests**. Then $\text{span}_{\mathbb{R}}\{\mathcal{F}_{\mathcal{L}}\}$ is an object in $\underline{\mathbb{R}\text{-Mod}}$, on which we declare **inner multiplication** of polynomials to obtain an algebra.

# Grading

**Graduation** $\mathrm{g}(\tau_1 \cdots \tau_k) = |\tau_1| + \ldots + |\tau_k|$ with $|\tau_i|$ being the number of vertices of the tree (the empty has zero vertices).

Setting $\mathcal{F}_n$ the set of forests of degree up to $n$:

$$\mathcal{A}_n\mathcal{T}_{\mathcal{L}} = \langle \mathcal{F}_n \rangle_{\underline{\mathbb{R}-\mathsf{Mod}}}$$

and

$$\mathcal{A}\mathcal{T}_{\mathcal{L}} = \coprod_{n=0}^{\infty} \mathcal{A}_n\mathcal{T}_{\mathcal{L}}$$

We have the inclusions

$$\mathcal{T}_{\mathcal{L}} \hookrightarrow \mathcal{F}_{\mathcal{L}} \hookrightarrow \mathcal{A}\mathcal{T}_{\mathcal{L}}.$$

# Dualizing: Co-Algebra

Define the **co-product** $\Delta : \mathcal{AT}_{\mathcal{L}} \to \mathcal{AT}_{\mathcal{L}} \otimes \mathcal{AT}_{\mathcal{L}}$ as $\underline{\mathbb{R}\text{-Alg}}$ morphism:

$$\Delta(\tau_1 \cdots \tau_k) := \Delta(\tau_1) \cdots \Delta(\tau_k) \quad \text{and} \quad \Delta(\mathbf{1}) := \mathbf{1} \otimes \mathbf{1}$$

recursively on generators via

$$\Delta(\tau) := \mathbf{1} \otimes \tau + \sum_{a \in \mathcal{L}} (B_+^a \otimes \mathrm{id})[\Delta(B_-^a(\tau))]$$

where $B_+^a(\mathbf{1}) = \bullet_a$ and $B_+^a(\tau_1 \cdots \tau_k) = [\tau_1, \ldots, \tau_k]_a$.
$B_-^a$ is its inverse, which removes the root it its label is $a$ and erases the entire tree otherwise, *i.e.*

$$B_-^a(B_+^b(\tau_1 \cdots \tau_k)) = \begin{cases} \tau_1 \cdots \tau_k & \text{if } a = b, \\ 0 & \text{otherwise.} \end{cases}$$

## Combinatorics via the Co-Product: “Topiary / Forestry”

By an **admissible cut** $c$ of a tree $\tau$ we mean detaching a set of branches from the tree.

Given a cut $c \in \tau$, denote by $R_c(\tau) \in \mathcal{T}_{\mathcal{L}}$ the remaining subtree and by $P_c(\tau) \in \mathcal{F}_L$ the forest of detached and newly planted branches.

For example: all cuts of the forest: 

Explicit description of the co-product in terms of cuts

$$\Delta(\tau) = \mathbf{1} \otimes \tau + \tau \otimes \mathbf{1} + \sum_{c \in \mathrm{C}(\tau)} R_c(\tau) \otimes P_c(\tau)$$

# Section 2

## Increments

# Increments

$T > 0$, $\mathbf{V}$ a vector-space, $k \in \mathbb{N}$, $k \geq 1$.

$$\mathcal{C}_k(\mathbf{V}) := \left\{ f \in \mathbf{C}([0,T]^k, \mathbf{V}) \ : \ f_{t_1,\dots,t_k} = 0 \text{ if } t_i = t_{i+1},\ 1 \leq i \leq k-1 \right\}$$

Space of **k-increments**, define $\mathcal{C}_* = \{\mathcal{C}_k : k \in \mathbb{N}\}$ in $\underline{\mathbb{N}\text{-grad-Mod}}$.

The **Co-Boundary** map

$$\delta : \mathcal{C}_k \to \mathcal{C}_{k+1} \qquad g \mapsto (\delta g)_{t_1 \cdots t_{k+1}} = \sum_{i=1}^{k+1} (-1)^i g_{t_1 \cdots \hat{t}_i \cdots t_{k+1}}$$

turns $(\mathcal{C}_*, \delta)$ into a long exact sequence.

Define the space of $k$-**cocycles**

$$\mathcal{Z}\mathcal{C}_k = \ker(\delta) \cap \mathcal{C}_k.$$

# Grading / Exterior Product

On $(\mathcal{C}_*, \delta)$ we have an **exterior product**:

$$g \in \mathcal{C}_n, \;\; h \in \mathcal{C}_m$$

then $gh \in \mathcal{C}_{m+n-1}$ defined by

$$(gh)_{t_1,\dots t_{m+n-1}} = g_{t_1,\dots,t_n} h_{t_n,\dots,t_{n+m-1}}$$

Then $\delta$ acts as **graded derivation**, in particular for $f, g \in \mathcal{C}_2$:

$$\delta(fg) = (\delta f)g - f(\delta g)$$

# An Important Example

Iterated integrals against smooth functions:

For $f \in \mathbf{C}^\infty([0,T],\mathbb{R}) \subset \mathcal{C}_1$, and $h \in \mathcal{C}_2$, let

$$\mathcal{I}(df\ \ h)_{ts} := \int_s^t h_{us} df_u \in \mathcal{C}_2$$

## Lemma

*Let $h \in \mathcal{C}_2$ such that $\delta h_{tus} = \sum_{i=1}^N h^{1,i}_{tu} h^{2,i}_{us}$ for some $N \in \mathbb{N}$, $h^{1,i}, h^{2,i} \in \mathcal{C}_2$ and let $x \in \mathbf{C}^\infty([0,T],\mathbb{R})$. Then*

$$\delta\mathcal{I}(dx\ \ h)_{tus} = \mathcal{I}(dx)_{tu} h_{us} + \sum_{i=1}^N \mathcal{I}(dx\ \ h^{1,i})_{tu} h^{2,i}_{us}$$

# Iterated Integrals and Rooted Trees

Let $\mathcal{L} = \{1, \dots, d\}$ and $x = \{x^a\}_{a \in \mathcal{L}} \subset \mathbf{C}^\infty([0,T])$.

Define the map

$$X : \mathcal{T}_{\mathcal{L}} \to \mathbf{C}([0,T]^2)$$

via its value on generators

$$(t,s) \mapsto X_{ts}^{\bullet_a} := \int_s^t dx_u^a = (\delta x^a)_{ts},$$

$$(t,s) \mapsto X_{ts}^{[\tau_1 \cdots \tau_k]_a} := \int_s^t \prod_{i=1}^k X_{us}^{\tau_i} dx_u^a$$

# Extension to a Morphism of Algebras

On $\mathcal{C}_2$ we have an $\mathbb{R}$-Alg structure, with **inner product**

$$\circ : \mathcal{C}_2 \otimes \mathcal{C}_2 \to \mathcal{C}_2 \quad f \otimes g \mapsto (f \circ g)_{ts} := f_{ts} g_{ts}$$

Freely adjoin unit $\mathcal{C}_2^+ = \mathcal{C}_2 \oplus e$, with $e_{st} = 1$ for all $s, t \in [0, T]$.

Extend $X$ to a morphism of $\mathbb{R}$-Alg: $X : \mathcal{AT}_{\mathcal{L}} \to \mathcal{C}_2^+$.

$$(t, s) \mapsto X_{ts}^{[\tau_1 \cdots \tau_k]_a} := \int_s^t X_{us}^{\tau_1} \circ \ldots \circ X^{\tau_k} dx_u^a.$$

And to the tensor product $\mathcal{AT}_{\mathcal{L}} \otimes \mathcal{AT}_{\mathcal{L}}$ via the exterior product $\mathcal{C}_2 \otimes \mathcal{C}_2 \to \mathcal{C}_3$

$$X^{\tau \otimes \sigma} \mapsto X^\tau \otimes X^\sigma \mapsto X^\tau X^\sigma$$

# Integration on a Sub-Algebra

Recall the family of integration maps associated to $x = \{x^a\}_{a\in\mathcal{L}} \subset \mathbf{C}^\infty([0,T])$

$$I^a : \mathcal{C}_2 \to \mathcal{C}_2, \qquad h \mapsto \mathcal{I}(x^a\ h)$$

As a consequence of the definitions we obtain the following fundamental relation

$$I^a(X^\sigma) = X^{[\sigma]_a} = X^{B_+^a(\sigma)}$$

Denote by $\mathcal{A}_X \subset \mathcal{C}_2^+$ the subalgebra generated by $\{X^\tau\}_{\tau\in\mathcal{T}_\mathcal{L}}$. Then the map $B_+^a$ represents integration on the subalgebra.

# Chen's Multiplicative Property I

From the family $\{x^a\}_{a\in\mathcal{L}}$ define **iterated integrals** recursively:

$$\mathcal{I}(dx^{a_1}\ dx^{a_2}\cdots dx^{a_n}) = \mathcal{I}(dx^{a_1}\ \mathcal{I}(dx^{a_2}\ dx^{a_3}\cdots dx^{a_n}))$$

The sub-algebra $\mathcal{A}_X$ contains these iterated integrals, which correspond to trees of the form $\sigma = [\cdots[\bullet_{a_n}]_{a_{n-1}}\cdots]_{a_1}$.

$$\mathcal{I}(dx^{a_1}\cdots dx^{a_n}) = I^{a_1}\cdots I^{a_{n-1}}(\delta x^{a_n}) = X^{B_+^{a_1}\cdots B_+^{a_{n-1}}(\bullet_{a_n})} = X^{[\cdots[\bullet_{a_n}]_{a_{n-1}}\cdots]_{a_1}}$$

From the action of $\delta$ on the integral we recover **Chen's multiplicative property**

$$\delta X^{\sigma} = \delta\mathcal{I}(dx^{a_1}\cdots dx^{a_n})_{stu} = \sum_{k=1}^{n-1}\mathcal{I}(dx^{a_1}\cdots dx^{a_k})_{st}\mathcal{I}(dx^{a_{k+1}}\cdots dx^{a_n})_{tu}$$

# Chen's Multiplicative Property II

Non-trivial cuts of $\sigma = [\cdots[\bullet_{a_n}]_{a_{n-1}}\cdots]_{a_1}$ break it into two pieces,

$$\Delta'(\sigma) = \sum_{k=1}^{n-1} [\cdots[\bullet_{a_k}]_{a_{k-1}}\cdots]_{a_1} \otimes [\cdots[\bullet_{a_n}]_{a_{n-1}}\cdots]_{a_{k+1}}$$

and hence

$$X^{\Delta'(\sigma)} = \sum_{k=1}^{n-1} X^{[\cdots[\bullet_{a_k}]_{a_{k-1}}\cdots]_{a_1}} X^{[\cdots[\bullet_{a_n}]_{a_{n-1}}\cdots]_{a_{k+1}}}$$

so that with Chen's multiplicative property

$$\delta X^{\sigma} = X^{\Delta'(\sigma)}$$

for all 'sticks' $\sigma$.

# Tree Multiplicative Property I

We can extend this fundamental commutativity property.

## Theorem

*The morphism $X : \mathcal{AT}_{\mathcal{L}} \to \mathcal{C}_2$ satisfies the relation:*

$$\delta X^{\tau} = X^{\Delta'(\tau)} \quad \text{for all } \tau \in \mathcal{AT}_{\mathcal{L}},$$

*i.e. the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{AT}_{\mathcal{L}} & \xrightarrow{\Delta'} & \mathcal{AT}_{\mathcal{L}} \otimes \mathcal{AT}_{\mathcal{L}} \\ \downarrow X & & \downarrow X \\ \mathcal{C}_2 & \xrightarrow{\delta} & \mathcal{C}_3 \end{array}$$

*where $\Delta'(\tau) = \Delta(\tau) - 1 \otimes \tau - \tau \otimes 1$ is the reduced co-product.*

# Tree Multiplicative Property II

Strategy of the proof:

1. Reduce to monomials (Forests) by using linearity.
2. Induction on degree $n$ of monomials.
3. Products of monomials, each of lower degree for which induction hypothesis holds:
   Requires understanding action $\Delta'(\tau\sigma)$ and $\delta X^{\tau\sigma}$.
4. Show relation for trees of degree $n$.

For the time being, we will only do step 4 (the interesting one).

## Proof of Step 4.

In this step it remains to prove the relation for a single tree of degree $n$, i.e. $\tau = [\tau_1 \cdots \tau_k]_a$. Write $\Delta'(\tau_1 \cdots \tau_k) = \sum_i \theta_i^1 \otimes \theta_i^2$. Since $|\tau_1 \cdots \tau_k| = n - 1$, by hypothesis

$$\delta X^{\tau_1 \cdots \tau_k} = X^{\Delta'(\tau_1 \cdots \tau_k)} = \sum_i X^{\theta_i^1} X^{\theta_i^2}$$

**Using the action of $\delta$ on $\mathcal{I}$ from the lemma:**

$$\begin{aligned}\delta X^{[\tau_1 \cdots \tau_k]_a} &= \delta \mathcal{I}(dx^a \; X^{\tau_1 \cdots \tau_k}) = \delta x^a X^{\tau_1 \cdots \tau_k} + \sum_i \mathcal{I}(dx^a \; X^{\theta_i^1}) X^{\theta_i^2} \\ &= X^{\bullet_a} X^{\tau_1 \cdots \tau_k} + \sum_i X^{[\theta_i^1]_a} X^{\theta_i^2} = X^{\bullet_a \otimes \tau_1 \cdots \tau_k} + \sum_i X^{[\theta_i^1]_a \otimes \theta_i^2} \\ &= X^{\Delta'([\tau_1 \cdots \tau_k]_a)}\end{aligned}$$

The last equality can be understood in terms of cuts. □

## Example

In one dimension forests of degree less or equal to three are:

$$\bullet,\ [\bullet],\ \bullet\bullet,\ [[\bullet]],\ \bullet[\bullet],\ \bullet\bullet\bullet,\ [\bullet\bullet]$$

The reduced co-product acts as follows:

$$\Delta'[\bullet] = \bullet\otimes\bullet, \qquad \Delta'(\bullet\bullet) = 2\bullet\otimes\bullet$$

$$\Delta'[[\bullet]] = [\bullet]\otimes\bullet + \bullet\otimes[\bullet]$$

$$\Delta'(\bullet[\bullet]) = \bullet\otimes\bullet\bullet + \bullet\bullet\otimes\bullet + [\bullet]\otimes\bullet + \bullet\otimes[\bullet]$$

$$\Delta'(\bullet^3) = 3\bullet^2\otimes\bullet + 3\bullet\otimes\bullet^2, \qquad \Delta'[\bullet\bullet] = \bullet\otimes\bullet\bullet + 2[\bullet]\otimes\bullet$$

Hence for example

$$\delta X^{[[\bullet],[\bullet]]} = \delta X^{[\bullet\bullet]} = X^{\bullet}X^{\bullet\bullet} + 2X^{[\bullet]}X^{\bullet}$$

# Section 4

## Regularity of $X$

# Topologizing 2- and 3-Increments

Let $\mu > 0$. For $f \in \mathcal{C}_2$ set

$$\|f\|_\mu := \sup_{s \neq t, s,t \in [0,T]} \left\{ \frac{f_{st}}{|s-t|^\mu} \right\}$$

and for $h \in \mathcal{C}_3$ we set

$$\|h\|_{\gamma,\rho} := \sup_{s,u,t \in [0,T]} \left\{ \frac{|h_{tus}|}{|u-s|^\gamma |t-u|^\rho} \right\}$$

$$\|h\|_\mu := \inf_{0<\rho_i<\mu} \left\{ \sum_{i=1}^N \|h_i\|_{\rho_i, \mu-\rho_i} \ : \ h = \sum_{i=1}^N h_i, \ h_i \in \mathcal{C}_3, \ N \in \mathbb{N} \right\}.$$

Define $\mathcal{C}_2^\mu := \{f \in \mathcal{C}_2 : \|f\|_\mu < \infty\}$, $\mathcal{C}_3^\mu := \{f \in \mathcal{C}_3 : \|f\|_\mu < \infty\}$, and finally $\mathcal{C}_k^{1+} = \cup_{\mu>1} \mathcal{C}_k^\mu$.

# The Splitting-Map of the Short Exact Sequence

## Theorem (The Λ-map)

*There exists a unique linear map* $\Lambda : \mathcal{Z}\mathcal{C}_3^{1+} \to \mathcal{C}_2^{1+}$

$$\delta\Lambda = \mathrm{id}_{\mathcal{Z}\mathcal{C}_3}.$$

*For any* $\mu > 1$, *this map is continuous from* $\mathcal{Z}\mathcal{C}_3^{\mu}$ *to* $\mathcal{C}_2^{\mu}$

$$\|\Lambda h\|_\mu \leq \frac{1}{2^\mu - 2}\|h\|_\mu, \qquad h \in \mathcal{Z}\mathcal{C}_3^{\mu}.$$

The map provides a splitting that we will repeatedly use.

$$0 \longrightarrow \mathcal{Z}\mathcal{C}_2^{1+} \xhookrightarrow{\text{incl}} \mathcal{C}_2^{1+} \underset{\Lambda}{\overset{\delta_{2\to3}}{\rightleftarrows}} \mathcal{Z}\mathcal{C}_3^{1+} \longrightarrow 0$$

# An Axiomatic Definition of the Integral

We can abstract the previous constructions by distilling only the properties of the integration maps $\{I^a : \mathcal{C}_2 \to \mathcal{C}_2\}$ that we needed.

## Definition

Call a linear map $I : \mathcal{D}_I \to \mathcal{D}_I$ on a sub-algebra $\mathcal{D}_I \subset \mathcal{C}_2^+$ containing the unit $e \in \mathcal{C}_2$ an **integral** if is satisfies the following properties.

1. $I(hf)_{ts} = I(h)_{ts} f_s$, for all $h \in \mathcal{D}_I, f \in \mathcal{C}_1$ where $(hf)_{ts} = h_{ts} f_s$,
2. $\delta I(h)_{tus} = I(e)_{tu} h_{us} + \sum_{i=1}^N I(h^{1,i})_{tu} h^{2,i}_{us}$ whenever $h \in \mathcal{D}_I$ and $\delta h_{tus} = \sum_{i=1}^N h^{1,i}_{tu} h^{2,i}_{us}$ for some $n \in \mathbb{N}$, $h^{1,i} \in \mathcal{D}_I$.

With this definition we can construct a homomorphism $X : \mathcal{AT}_{\mathcal{L}} \to \mathcal{C}_2$ as before satisfying the commutativity relation.

# Regularity and Branched Rough Paths

Given $\gamma \in (0,1]$ define $q_\gamma$ on trees as

$$q_\gamma(\tau) = \begin{cases} 1 & \text{if } |\tau| \leq 1/\gamma \\ \frac{1}{2^{\gamma|\tau|}-2} \sum q_\gamma(\tau^{(1)}) q_\gamma(\tau^{(2)}) & \text{if } |\tau| > 1/\gamma \end{cases}$$

The splitting stems from the the reduced co-product. On forests $\tau = \tau_1 \cdots \tau_k$, set $q_\gamma(\tau) = q_\gamma(\tau_1) \cdots q_\gamma(\tau_k)$.

## Definition

Let $\gamma > 0$. We call a morphism of algebras $X : \mathcal{AT}_{\mathcal{L}} \to \mathcal{C}_2$ a $\gamma$-**branched rough path** ($\gamma$-BRP) if it satisfies $\delta X = X^{\Delta'}$ and

$$\|X^\tau\|_{\gamma|\tau|} \leq BA^{|\tau|} q_\gamma(\tau), \quad \text{for all } \tau \in \mathcal{F}_{\mathcal{L}}$$

and constants $B \in [0,1]$ and $A \geq 0$.

# Extension from a Finite Set of Trees I

**Theorem**

*Let $X : \mathcal{A}_n\mathcal{T}_{\mathcal{L}} \to \mathcal{C}_2$ be a given morphism satisfying $\delta X = X^{\Delta'}$ and suppose that there exist $\gamma > 0$, $A \geq 0$, $B \in [0,1]$ such that*

$$\|X^\tau\|_{\gamma|\tau|} \leq BA^{|\tau|} q_\gamma(\tau) \quad \text{for all } \tau \in {\mathcal{T}_{\mathcal{L}}}^n,$$

*with $\gamma(n+1) > 1$. Then there exists a unique extension of $X$ to $\mathcal{A}\mathcal{T}_{\mathcal{L}}$ as a $\gamma$-branched rough path with the same bounds.*

# Extension from a Finite Set of Trees II

Outline of Proof: Via Induction and using the diagram below.

1. Show that $X^{\Delta'}$ maps to $\mathcal{Z}\mathcal{C}_3^{|\tau|\gamma}$ for "large trees" $\tau$.
2. Use continuity of $\Lambda$ to show bounds for $X^\tau$ via splitting of short exact sequence.

$$\begin{array}{ccc}
\mathcal{A}\mathcal{T}_{\mathcal{L}} & \xrightarrow{\Delta'} & \mathcal{A}\mathcal{T}_{\mathcal{L}} \otimes \mathcal{A}\mathcal{T}_{\mathcal{L}} \\
\downarrow X & & \downarrow X \\
\mathcal{C}_2 & \xrightarrow{\delta} & \mathcal{C}_3 \\
\uparrow \text{incl} & & \uparrow \text{incl} \\
\mathcal{C}_2^{\gamma|\tau|} & \xleftarrow{\Lambda} & \mathcal{Z}\mathcal{C}_3 \cap \mathcal{C}_3^{\gamma|\tau|}
\end{array}$$

# Extension from a Finite Set of Trees III

## Sketch of Proof.

Assume that we have a bounded extension $X : \mathcal{A}_m\mathcal{T}_{\mathcal{L}} \to \mathcal{C}_2$ satisfying commutativity. (True for $n = m$). For the induction step: Since $\gamma m \geq \gamma(n+1) > 1$, we have for $|\tau| = m$

$$\|X^{\Delta'(\tau)}\|_{m\gamma} \leq \sum_i' \|X^{\tau_i^{(1)}\otimes\tau_i^{(2)}}\|_{m\gamma} \leq \sum_i' \|X^{\tau_i^{(1)}}\|_{|\tau_i^{(1)}|\gamma} \|X^{\tau_i^{(2)}}\|_{|\tau_i^{(2)}|\gamma} < \infty$$

and

$$\delta X^{\Delta'(\tau)} = \sum_i' [\delta X^{\tau_i^{(1)}}] X^{\tau_i^{(2)}} - X^{\tau_i^{(1)}} [\delta X^{\tau_i^{(2)}}] = \sum_i' X^{(\mathrm{id}\otimes\Delta' - \Delta'\otimes\mathrm{id})\Delta'(\tau)} = 0$$

Thus $X^{\Delta'(\tau)} \in \mathcal{Z}\mathcal{C}_3 \cap \mathcal{C}_3^{m\gamma} = \mathcal{Z}\mathcal{C}_3^{m\gamma}$. Now using continuity of $\Lambda$ and splitting to get $\|X^\tau\|_{\gamma|\tau|} = \|\Lambda X^{\Delta'(\tau)}\|_{\gamma|\tau|} \leq B^2 A^{|\tau|} q_\gamma(\tau)$. □

# Section 5

## Weakly Controlled Paths

We want to give a sensible notion of solutions of **rough differential equations**

$$\delta y = \sum_{a \in \mathcal{L}} I^a(f_a(y)), \qquad y_0 = \eta \in \mathbb{R}^k$$

where $I^a$ is a family of integration maps giving rise to a $\gamma - BRP$, $f_a$ is a collection of (sufficiently regular) vector-fields.

## Definition

Let $X$ be a $\gamma$-BRP and $n$ the largest integer such that $n\gamma \leq 1$. For $\kappa \in (1/(n+1), \gamma]$, the path $y : [0, T] \to \mathbb{R}$ is a $\kappa$-**weakly controlled** by $X$ if there exists $\{y^\tau \in \mathcal{C}_1^{|\tau|\kappa}\}_{\tau \in \mathcal{F_L}^{n-1}}$ and remainders $\{y^\sharp \in \mathcal{C}_2^{n\kappa}, y^{\tau,\sharp} \in \mathcal{C}_2^{(n-|\tau|)\kappa}\}_{\tau \in \mathcal{F_L}^{n-1}}$ such that

$$\delta y = \sum_{\tau \in \mathcal{F_L}^{n-1}} X^\tau y^\tau + y^\sharp \tag{1}$$

$$\delta y^\tau = \sum_{\sigma \in \mathcal{F_L}^{n-1}} \sum_{\rho} c'(\sigma, \tau, \rho) X^\rho y^\sigma + y^{\tau,\sharp} \tag{2}$$

for $\tau \in \mathcal{F_L}^{n-1}$, with $\delta y^\tau = y^{\tau,\sharp}$ when $|\tau| = n-1$. Let $\mathcal{Q}_\kappa(X)$ be the vector space of $\kappa$-weakly controlled paths with norm $\|\cdot\|_{\mathcal{Q},\kappa}$

$$\|y\|_{\mathcal{Q},\kappa} = |y_0| + \|y^\sharp\|_{n\kappa} + \sum_{\tau \in \mathcal{F_L}^{n-1}} \|y^{\tau,\sharp}\|_{\kappa(n-|\tau|)}.$$

## Example

Let us give an example with $d = 1$ of the structure of a controlled path. Take $\gamma > 1/5$ so that $n = 4$. Then $y \in \mathcal{Q}_\gamma$ corresponds to the set of paths

$$y \in \mathcal{C}_1^{\gamma}, \quad y^{\bullet} \in \mathcal{C}_1^{\gamma}, \quad y^{[\bullet]}, y^{\bullet\bullet} \in \mathcal{C}_1^{2\gamma}, \quad y^{[\bullet\bullet]}, y^{[\bullet]\bullet} y^{[\bullet\bullet]}, y^{[[\bullet]]}, y^{\bullet\bullet\bullet} \in \mathcal{C}_1^{3\gamma}$$

## Example (Continued)

And the following algebraic relations hold

$$\delta y = X^{\bullet} y^{\bullet} + X^{[\bullet]} y^{[\bullet]} + X^{\bullet\bullet} y^{\bullet\bullet} + X^{[\bullet\bullet]} y^{[\bullet\bullet]} + X^{[\bullet]\bullet} y^{[\bullet]\bullet} + X^{[\bullet\bullet]} y^{[\bullet\bullet]} + \\ + X^{\bullet\bullet\bullet} y^{\bullet\bullet\bullet} + X^{[[\bullet]]} y^{[[\bullet]]} + y^{\sharp}$$

$$\delta y^{\bullet} = X^{\bullet}(y^{[\bullet]} + 2y^{\bullet\bullet}) + X^{[\bullet]}(y^{[[\bullet]]} + y^{[\bullet]\bullet}) + X^{\bullet\bullet}(y^{[\bullet]\bullet} + y^{[\bullet\bullet]} + 3y^{\bullet\bullet\bullet}) + y^{\bullet,\sharp}$$

$$\delta y^{[\bullet]} = X^{\bullet}(y^{[\bullet]\bullet} + 2y^{[\bullet\bullet]} + y^{[[\bullet]]}) + y^{[\bullet],\sharp}$$

$$\delta y^{\bullet\bullet} = X^{\bullet}(y^{[\bullet]\bullet} + y^{\bullet\bullet\bullet}) + y^{\bullet\bullet,\sharp}$$

$$\delta y^{[\bullet\bullet]} = y^{[\bullet\bullet],\sharp} \qquad \delta y^{[\bullet]\bullet} = y^{[\bullet]\bullet,\sharp}$$

$$\delta y^{\bullet\bullet\bullet} = y^{\bullet\bullet\bullet,\sharp} \qquad \delta y^{[[\bullet]]} = y^{[[\bullet]],\sharp}$$

with remainders of orders

$$y^{\sharp} \in \mathcal{C}_2^{4\gamma}, \quad y^{\bullet,\sharp} \in \mathcal{C}_2^{3\gamma}, \quad y^{[\bullet],\sharp}, y^{\bullet\bullet,\sharp} \in \mathcal{C}_2^{2\gamma} \quad y^{[\bullet\bullet],\sharp}, y^{[\bullet]\bullet,\sharp}, y^{\bullet\bullet\bullet,\sharp}, y^{[[\bullet]],\sharp} \in \mathcal{C}_2^{\gamma}.$$

# Properties of Weakly Controlled Paths

- An element in $\mathcal{Q}_\kappa(X)$ is a path together with all its increments $\{y^\tau\}$ and an expansion in terms of $X$ with remainder $y^\sharp$.
- Coefficients of this expansion have similar expansions of lower degree.
- The space $\mathcal{Q}_\kappa(X)$ can be endowed with the structure of a $\mathbb{R} - Algebra$.
- It is closed under composition with sufficiently regular functions.

# Closedness under composition with regular functions

Let $\mathcal{L}_1 = \{1, \ldots, k\}$ and $\mathcal{I}\mathcal{L}_1 = \cup_{m\geq 0}\mathcal{L}_1^m$ the set of **multiindices**, with $|\overline{b}| = n$ whenever $\overline{b} \in \mathcal{L}_1^n$.

## Lemma

*Let $n$ the largest integer such that $n\gamma \leq 1$, $\varphi \in C_b^n(\mathbb{R}^k, \mathbb{R})$ and $y \in \mathcal{Q}_\kappa(X; \mathbb{R}^k)$, then $z_t = \varphi(y_t)$ is a weakly controlled path, $z \in \mathcal{Q}_\kappa(X; \mathbb{R})$ where its coefficients are given by*

$$z^\tau = \sum_{m=1}^{n-1} \sum_{\substack{\overline{b}\in\mathcal{I}\mathcal{L}_1 \\ |\overline{b}|=m}} \frac{\varphi_{\overline{b}}(y)}{m!} \sum_{\substack{\tau_1,\ldots,\tau_m\in{\mathcal{F}_\mathcal{L}}^{n-1} \\ \tau_1\cdots\tau_m=\tau}} y^{\tau_1,b_1}\cdots y^{\tau_m,b_m}, \qquad \tau \in {\mathcal{F}_\mathcal{L}}^{n-1}$$

*(note that all the sums are finite).*

## Sketch of Proof.

Taylor expand $\varphi$ to get $(\delta\varphi)_{\xi'\xi}$:

$$\varphi(\xi') - \varphi(\xi) = \sum_{m=1}^{n-1} \sum_{\substack{\overline{b}\in\mathcal{IL}_1\\ |\overline{b}|=m}} \frac{\varphi_{\overline{b}}(\xi)}{m!}(\xi'-\xi)^{\overline{b}} + O(|\xi'-\xi|^n)$$

thus

$$\delta z_{ts} = \sum_{m=1}^{n-1} \sum_{\substack{\overline{b}\in\mathcal{IL}_1\\ |\overline{b}|=m}} \frac{\varphi_{\overline{b}}(y_s)}{m!}(\delta y_{ts})^{\overline{b}} + O(|t-s|^{n\kappa})$$

$$= \sum_{m=1}^{n-1} \sum_{\tau^1\cdots\tau^m\in\mathcal{F}_{\mathcal{L}}^{n-1}} \sum_{\substack{\overline{b}\in\mathcal{IL}_1\\ |\overline{b}|=m}} \frac{\varphi_{\overline{b}}(y_s)}{m!} y_s^{\tau^1 b_1}\cdots y_s^{\tau^m b_m} X_{ts}^{\tau^1\cdots\tau^m} + O(|t-s|^{n\kappa})$$

Also every $z^\tau$ has to satisfy the $\delta$-equations: details skipped.

## Extending the Integration maps

Recall the family of integrals $\{I^a : \mathcal{D}_I \to \mathcal{D}_I\}_a$ (either defined axiomatically or as integration against smooth functions).

We can extend their domain to $\mathcal{C}_1$, *viz.*
Embed $f \in \mathcal{C}_1 \mapsto f_s e_{st} \in \mathcal{C}_2^+$, then set

$$I(f) = I(fe)$$

and since $fe = ef + \delta f$ we have

$$I(f) = I(e)f + I(\delta f)$$

for any $f \in \mathcal{C}_1$ such that $\delta f \in \mathcal{D}_2$

**Theorem**

*The integral maps $\{I^a\}_{a\in\mathcal{L}}$ can be extended to maps $I^a : \mathcal{Q}_\kappa(X) \to \delta\mathcal{Q}_\kappa(X)$. If $y \in \mathcal{Q}_\kappa(X)$ then $\delta z = I^a(y)$ is such that*

$$\delta z = X^{\bullet_a} z^{\bullet_a} + \sum_{\tau\in\mathcal{T}_{\mathcal{L}}{}^n} X^\tau z^\tau + z^\flat \tag{3}$$

*where $z^{\bullet_a} = y$, $z^{[\tau]_a} = y^\tau$ and zero otherwise. Moreover*

$$z^\flat = \Lambda\left[\sum_{\tau\in\mathcal{F}_{\mathcal{L}}{}^{n-1}\cup\{\mathbf{1}\}} X^{B_a^+(\tau)} y^{\tau,\sharp}\right] \in \mathcal{C}_2^{\kappa(n+1)}.$$

# Proof I

Recall $I^a(y) = I^a(e)y + I^a(\delta y)$, hence we are done once we can show that $I^a(\delta y)$ is well-defined.

Since $y \in \mathcal{Q}_k$, we have the expansion

$$\delta y = \sum_{\tau \in \mathcal{F}_{\mathcal{L}}{}^{n-1}} X^\tau y^\tau + y^\sharp$$

Since $\mathcal{D}_I$ is a linear space, we have $\sum_{\tau \in \mathcal{F}_{\mathcal{L}}{}^{n-1}} X^\tau y^\tau \in \mathcal{D}_I$, so that

$$I^a\Big(\sum_{\tau \in \mathcal{F}_{\mathcal{L}}{}^{n-1}} X^\tau y^\tau\Big) = \sum_{\tau \in \mathcal{F}_{\mathcal{L}}{}^{n-1}} I^a(X^\tau) y^\tau = \sum_{\tau \in \mathcal{F}_{\mathcal{L}}{}^{n-1}} X^{[\tau]_a} y^\tau = I^a(\delta y - y^\sharp)$$

Hence we will be done if we can show that $I^a(y^\sharp)$ is well defined.

# Proof II

Strategy: show that $\delta I^a(y^\sharp) \in \mathcal{ZC} \cap \mathcal{C}_3^{(n+1)\kappa} \subset \mathcal{ZC}_3^{1+}$ and hence in the domain of $\Lambda$: uses axiomatic properties of $I^a$ via

$$\delta I^a(y^\sharp) = I^a(e)y^\sharp + \sum_{\tau \in \mathcal{F}_{\mathcal{L}}{}^{n-1}} I^a(X^\tau)y^{\tau,\sharp} = X^{\bullet_a}y^\sharp + \sum_{\tau \in \mathcal{F}_{\mathcal{L}}{}^{n-1}} X^{[\tau]_a}y^{\tau,\sharp}$$

R.H.S. are well defined and well behaved objects. We need a technical lemma to calculate $\delta y^{\tau,\sharp}$, norm-estimates and properties of derivation.

Now set $I^a(y^\sharp) = \Lambda\left[X^{\bullet_a}y^\sharp + \sum_{\tau \in \mathcal{F}_{\mathcal{L}}{}^{n-1}} X^{[\tau]_a}y^{\tau,\sharp}\right]$

# Proof III

Now combine everything:

$$I^a(y) = I^a(e)y + I^a(\delta y) = X^{\bullet_a} y + I^a\Big( \sum_{\tau \in \mathcal{F}_{\mathcal{L}}{}^{n-1}} X^\tau y^\tau \Big) + \Lambda[\ldots]$$

but this is just

$$I^a(y) = X^{\bullet_a} y + \sum_{\tau \in \mathcal{F}_{\mathcal{L}}{}^{n-1}} X^{[\tau]_a} y^\tau + \Lambda[\ldots]$$

with $\Lambda[\ldots] \in \mathcal{C}_2^{\kappa(n+1)}$, as claimed.

# Rough Differential Equations I

Let $\{f_a\}_{a=1,\dots d} \subset \mathbf{CB}^n(\mathbb{R}^k;\mathbb{R}^k)$ be vector-fields, where $n$ is the largest integer such that $n\gamma \leq 1$. Given integral maps $I^a$ which defining a $\gamma$-BRP $X$ the **rough differential equation**

$$\delta y = \sum_{a\in\mathcal{L}} I^a(f_a(y)), \qquad y_0 = \eta \in \mathbb{R}^k \tag{4}$$

in the time interval $[0, T]$.

- Previous lemma showed that $f_a(y)$ is a $\kappa$-weakly controlled, whenever $y$ is.
- Previous theorem showed that we can integrate $\kappa$-weakly controlled paths against $I^a$, obtaining a $\kappa$ weakly controlled 2-increment.

Thus it makes sense to speak of a solution $y \in \mathcal{Q}_\gamma(X;\mathbb{R}^k)$ via a fixed point problem in $\mathcal{Q}_\gamma(X;\mathcal{R}^k)$ of

$$\delta\Gamma(y) = \sum_{a\in\mathcal{L}} I^a(f_a(y)), \qquad \Gamma(y)_0 = \eta$$

# Rough Differential Equations II

**Theorem**

*If $\{f_a\}_{a\in\mathcal{L}}$ is a family of $C_b^n$ vectorfields then the rough differential equation $\delta y$ has a global solution $y \in \mathcal{Q}_\gamma(X;\mathcal{R}^k)$ for any initial condition $\eta \in \mathbb{R}^k$.*
*If the vectorfields are $C_b^{n+1}$ the solution $\Phi(\eta, X) \in \mathcal{Q}_\gamma(X;\mathbb{R}^k)$ is unique and the map $\Phi : \mathbb{R}^k \times \Omega^\gamma_{\mathcal{T}_\mathcal{L}} \to \mathcal{Q}_\gamma(X;\mathbb{R}^k)$ is Lipschitz in any finite interval $[0, T]$.*

# Summary

- By endowing the set of rooted decorated trees with algebraic structure, we obtained a **multiplicative property**.
- It uses the combinations of trees and algebraic integration theory to **define path wise integration against integrands with roughness** $\gamma > 0$.
- This theory can be used to study controlled and rough differential equations.